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LETTER TO THE EDITOR

The fractional-graded extension of the Virasoro algebra†

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Abstract. Extension of the Virasoro algebra by the infinite number of generators with homogeneous fractional gradation and the conformal dimensions $1 < \Delta \leqslant 2$ is given. We obtain the new families of centrally extended, constrained algebras, describing the conformal series of dimension $[(k/N)+1]$, $N \in \mathbb{N}$ and $k \in \{1, \ldots, N\}$.

It is well known that the conformal invariance in two dimensions plays a crucial role in the construction of the two-dimensional field-theoretic models [1] which are employed in the field-theoretic description of string models. $D = 2$ conformal symmetry algebra is infinite-dimensional and is described by the product of two copies of the Virasoro algebra [2].

We recall that the Virasoro algebra is a central extension of the complexification of the Lie algebra *Vect* S^1 of (real) vector fields on the circle S^1:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \qquad m, n \in \mathbb{Z} \tag{1}$$

where c denotes a central element such that $[L_n, c] = 0$. The systematic study of the properties of (1) are found, for example, in [3, 4].

The Virasoro algebra admits many extensions and generalizations, which lead to various interesting two-dimensional models—for example the bosonic W-algebras (W_N, w_∞, W_∞, $\hat{W}_\infty$), N-extended superalgebras, twisted affine conformal algebras, parafermion algebras etc. (For a review see, for example [5] and references therein).

The Virasoro generators can be described by the conformal field of the dimension 2. In the generalizations listed above one considers the additional set of the fields of other conformal dimensions (integer, half-integer or fractional) and finds the conditions of the associativity of the extended algebraic system. The rule selecting new generators is also governed by the requirement of additional symmetries (e.g. supersymmetries).

In this letter we propose the extension of the Virasoro algebra (1) by adding the infinite number of the generators $V^{k/N}_{m/M}$ with the fractional gradation of their conformal dimension to the Virasoro generators L_m. In this way, we get the infinite number of the families of the extended Virasoro algebras, parametrized by two integer parameters M and $N \in \mathbb{N}$.

The generalization to the multi-index parametrization of the Virasoro generators has also been considered earlier by Fairlie *et al* [6]. Our construction is, however, different than theirs because the fractional-graded Virasoro algebra is generated by the semi-direct

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product of the Virasoro generator L_m and the $N-1$ generators $V^{k/N}_{m/M}$. We would like to conclude therefore that the relation of our algebra to the $S\,Diff_0(S^2)$ symmetries is not clear.

Let $\mathcal{A}$ denote the set of the additional generators $V^{k/N}_{m/M}$ associated with the generators L_m of the Virasoro algebra (1). Let $m \in \mathbb{Z}$, M and $N \in \mathbb{N}$ and $k \in \{1, 2, \ldots, N\}$. We put $V^{N/N}_m = V^1_m \equiv L_m$. Let us introduce some binary operation $\langle , \rangle$ defined on the space $\mathcal{A}$, such that:

$$\langle \mathcal{A}, \mathcal{A} \rangle : \to \mathcal{A} \tag{2}$$

define the fractional extension of the Virasoro algebra, by the following relation:

$$\begin{aligned}\langle V^{k/N}_{m/M}, V^{l/N}_{n/M} \rangle := \frac{1}{MN}(lm - kn) V^{((k+l-N)/N)_+}_{(m+n)/M} + \frac{c}{12} m(m^2 - 1)\delta^{k,N}\delta^{l,N}\delta_{m,-n} \\ + \frac{1}{MN}(c_1 k - c_2) m(\delta^{k,l} - \delta^{k,N}\delta^{l,N})\delta_{m,-n}\end{aligned} \tag{3}$$

where $k, l \in \{1, 2, \ldots, N\}$, $m, n \in \mathbb{Z}$ and M, $N \in \mathbb{N}$. By the symbol $((k+l-N)/N)_+$ we denote the condition†

$$k + l - N > 0 \tag{4}$$

and c_1, c_2 are some constraints.

It is easy to see that the above formula (3) is equivalent to the following system:

$$\langle V^1_{m/M}, V^1_{n/M} \rangle = \frac{1}{M}(m-n)V^1_{(m+n)/M} + \frac{c}{12}\left[\left(\frac{m}{M}\right)^3 - \left(\frac{m}{M}\right)\right]\delta_{m,-n} \tag{5a}$$

$$\langle V^1_{m/M}, V^{\tilde{l}/N}_{n/M} \rangle = \frac{1}{MN}(\tilde{l}m - Nn)V^{\tilde{l}/N}_{(m+n)/M} \tag{5b}$$

$$\begin{aligned}\langle V^{\tilde{k}/N}_{m/M}, V^{\tilde{l}/N}_{n/M} \rangle = \frac{1}{MN}(\tilde{l}m - \tilde{k}n)V^{((k+l-N)/N)_+}_{(m+n)/M} \\ + \frac{1}{MN}(c_1\tilde{k} - c_2)m\delta^{\tilde{k},\tilde{l}}\delta_{m,-n} \qquad (\tilde{k}, \tilde{l} \in \{1, 2, \ldots, N-1\})\end{aligned} \tag{5c}$$

which we call the fractional-graded Virasoro algebra.

Proposition. The algebraic system (3) forms a Lie algebra, i.e. the operation $\langle , \rangle$ is skewsymmetric;

$$\langle V^{k/N}_{m/M}, V^{l/N}_{n/M} \rangle = -\langle V^{l/N}_{n/M}, V^{k/N}_{m/M} \rangle \tag{6}$$

and satisfies the Jacobi identity:

$$\langle V^{p/N}_{s/M}, \langle V^{k/N}_{m/M}, V^{l/N}_{n/M} \rangle\rangle + \langle V^{k/N}_{m/M}, \langle V^{l/N}_{n/M}, V^{p/N}_{s/M} \rangle\rangle + \langle V^{l/N}_{n/M}, \langle V^{p/N}_{s/M}, V^{k/N}_{m/M} \rangle\rangle = 0. \tag{7}$$

Proof. The skewsymmetry of (3) is obvious. It is easy to check, that for the classical part ($c = \tilde{c} = 0$) the Jacobi identity (7) is fulfilled, thus we prove only the existence of the second central term in (3).

† The condition (4) is equivalent to constraints imposed on the generators with the negative upper indices, i.e. $V^{k/N<0}_{m/M} = 0$.

Let us assume that the central extension of the algebra (3), has the form:

$$\langle \widetilde{V^{k/N}_{m/M}, V^{l/N}_{n/M}} \rangle = \langle V^{k/N}_{m/M}, V^{l/N}_{n/M} \rangle + cA^{k/N,l/N}_{m/M,n/M} \tag{8}$$

where

$$A^{k/N,l/N}_{m/M,n/M} \in \mathbb{C} \qquad \text{and} \qquad c \in \mathbb{C}.$$

Additionally, we assume that:

$$A^{k/N,l/N}_{m/M,n/M} = -A^{k/N,l/N}_{n/M,m/M} = -A^{l/N,k/N}_{n/M,m/M}. \tag{9}$$

From the Jacobi identity (7) we get the following 2-co-cycle equation:

$$\frac{c}{MN}[(lm - kn)A^{p/N,((k+l-N)/N)_+}_{s/M,(m+n)/M} + (pn - ls)A^{k/N,((l+p-N)/N)_+}_{m/M,(n+s)/M} + (ks - pm)A^{l/N,((p+k-N)/N)_+}_{n/M,(s+m)/M}] = 0. \tag{10}$$

If $p = k = l = N$ then from (10) we obtain the equation describing the standard 2-co-cycles of the Virasoro algebra (*5a*).

From the structure of the algebra (3) and Jacobi identity (7) it follows that:

$$p = N \qquad k = l \neq N \qquad \text{and} \qquad A^{1,((2k-N)/N)_+}_{s/M,(m+n)/M} = 0 \tag{11}$$

as well as $s = -m - n$. Thus, from (10) and (11) we get the 2-co-cycle equation of the form:

$$\frac{c}{MN}[(Nn + km + kn)A^{k/N,k/N}_{m/M,-m/M} - (Nm + km + kn)A^{k/N,k/N}_{n/M,-n/M}] = 0 \tag{12}$$

which has the following general solution:

$$A^{k/N,l/N}_{m/M,n/M} = \frac{1}{NM}(c_1 k \pm c_2) m \delta^{k/N,l/N} \delta_{m/M,-n/M}. \tag{13}$$

□

From the structure of the algebra (3) it follows that the 2-co-cycle (13) describe the non-trivial central terms, because it cannot be eliminated by the change of the basis $V^{k/N}_{m/M}$.

Indeed, consider the shift:

$$\tilde{V}^{k/N}_{m/M} = V^{k/N}_{m/M} - \hat{c}\delta_{m/M,0}. \tag{14}$$

If $k = N$ and $\hat{c} = c/24M$, then from (*5a*) we get the modified Virasoro algebra:

$$\langle \tilde{V}^1_{m/M}, \tilde{V}^1_{n/M} \rangle = \frac{1}{M}(m - n)\tilde{V}^1_{(m+n)/M} + \frac{c}{12}\left(\frac{m}{M}\right)^3 \delta_{m/M,-n/M}. \tag{15}$$

If $k \in \{1, 2, \ldots, N-1\}$ and $\hat{c} = c_1/2MN$, then from (*5c*) we have:

$$\langle \tilde{V}^{k/N}_{m/M}, \tilde{V}^{l/N}_{n/M} \rangle = \frac{1}{MN}(lm - kn)\tilde{V}^{((k+l-N)/N)_+}_{(m+n)/M} - \frac{c_2}{MN} m \delta^{k,l} \delta_{m,-n}. \tag{16}$$

Thus one can check that the central term of the form (c_2/MN) is always non-trivial.

Example. Let $N = 3$, then from (5) it follows that the 1/3-graded Virasoro algebra has the form:

$$\begin{aligned}
\langle V^1_{m/M}, V^1_{n/M}\rangle &= (m-n)V^1_{(m+n)/M} + \frac{c}{12}\left[\left(\frac{m}{M}\right)^3 - \frac{m}{M}\right]\delta_{m/M,-n/M} \\
\langle V^1_{m/M}, V^{1/3}_{n/M}\rangle &= \frac{1}{3M}(m-3n)V^{1/3}_{(m+n)/M} \\
\langle V^1_{m/M}, V^{2/3}_{n/M}\rangle &= \frac{1}{3M}(2m-3n)V^{2/3}_{(m+n)/M} \\
\langle V^{1/3}_{m/M}, V^{1/3}_{n/M}\rangle &= \frac{1}{3M}(c_1+c_2)m\delta_{m/M,-n/M} \\
\langle V^{2/3}_{m/M}, V^{2/3}_{n/M}\rangle &= \frac{2}{3M}(m-n)V^{1/3}_{(m+n)/M} + \frac{1}{3M}(2c_1+c_2)m\delta_{m/M,-n/M}.
\end{aligned} \tag{17}$$

Let us consider the properties of the algebra (3).

(i) If $N = M = 1$ then assuming that $k, l \in \{1, 2, 3, \ldots\}$, we obtain the w_∞ algebra considered in [7]:

$$\langle V^k_m, V^l_n\rangle = (lm - kn)V^{k+l-1}_{m+n} + \frac{c}{12}m(m^2-1)\delta^{k,1}\delta^{l,1}\delta_{m,-n}. \tag{18}$$

In this case the second central term in (3) disappear and the generator V^k_m has the conformal dimension $\Delta_k = k + 1$.

(ii) The new property of the fractional-graded algebra (3) is the existence of the two limits $N \to \infty$: if $k \to 1$ and $l \to 1$, then:

$$\lim_{N\to\infty}\langle V^{k/N}_{m/M}, V^{l/N}_{n/M}\rangle \to 0 \tag{19}$$

and if $k \to \infty$ as well as $l \to \infty$, then:

$$\lim_{N\to\infty}\langle V^{k/N}_{m/M}, V^{l/N}_{n/M}\rangle \to \langle V^1_{m/M}, V^1_{n/M}\rangle. \tag{20}$$

In the limit (20) the second central term in (3) depends on the value of the central charges c_1 and c_2. From the formula (19) it follows that the conformal dimensions $\Delta_{k,N}$ and $\Delta_{l,N}$ of the generators $V^{k/N}_{m/M}$ and $V^{l/N}_{n/M}$ are converging to the unity ($\Delta_{1,\infty} \to 1$), while from (20) we can see that $\Delta_{\infty,\infty} \to 2$. Thus, if the condition (4) is satisfied, the algebra (3) describes the families of the conformal densities with the dimensions:

$$\Delta_{k,N} = \left[\frac{k}{N} + 1\right] \in (1, 2) \tag{21}$$

where $N \in \mathbb{N}$ and $k \in \{1, 2, \ldots, N\}$.

These families are presented in table 1.

We note here that due to the condition (4) the gradation $1/M$ of the lower indices ('modes') m and n has a secondary meaning. It is only important when we study the representations of the algebra (3).

One can see that the dimensions (21) of conformal families (table 1) are different from the ones described in the parafermion theory. The $\mathbb{Z}_N$ parafermion algebra of Zamolodchikov and Fateev [8] contains $N-1$ parafermion currents ψ_m, $m = 1, 2, \ldots, N-1$.

Table 1. The conformal families of the generators of fractional-graded Virasoro algebra (3), for different N.

N	Conformal dimensions	Additional commentary
1	$2, 3, 4, \ldots$	w_∞ algebra if $k, l = 1, 2, 3, 4, \ldots$
	2	Virasoro algebra if $k = l = 1$
2	$\frac{3}{2}, 2$	$k, l = 1, 2$
3	$\frac{4}{3}, \frac{5}{3}, 2$	$k, l = 1, 2, 3$
4	$\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$	$k, l = 1, 2, 3, 4$
...	...	
$\to \infty$	$\to 1$	Kac-Moody type algebras if $k \to 1$ and $l \to 1$
	$\to 2$	Virasoro algebra if $k \to \infty$ and $l \to \infty$

It follows from the $\mathbb{Z}_N$ symmetry of the model that these currents are primary fields of the Virasoro algebra with conformal dimension $\Delta_m = m(N-m)/N$. Analogously, the conformal field theories based on a $G \otimes G/G$ coset space [9] do not contain the series of the type in table 1.

The main problem connected with the constrained algebra (3) is to find their highest weight representations [3] and the current algebra associated with the 'modes' algebra (3).

All of the standard parafermion theories start from the fractional current algebra, where the operator product expansion among the currents has the generic form:

$$\psi_i(z)\psi_j(w) = q_{ij}(z-w)^{-\Delta_i-\Delta_j}(1+\cdots) + \sum_k f_{ijk}(z-w)^{-\Delta_i-\Delta_j+\Delta_k}(\psi_k(w)+\cdots) \qquad (22)$$

where q_{ij} and f_{ijk} are the structure constants.

When fractional powers of $(z-w)$ appear in (22), some of the currents will necessarily have fractional dimensions. In this case, the algebra is non-local, due to the presence of Riemann cuts in the complex plane. This is the main reason why it is difficult to realize the 'modes' algebra (3) as the fractional current algebra, using the standard methods of conformal field theory. To this end, we need to find the systematic procedure of computation of the 'fractional modes algebra' from expressions of the type (22) and vice-versa (one can find some examples in [8, 10]).

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